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Introduction

- In this topic, we will
  - Discuss 2nd-order initial-value problems (IVPS)
  - Explain how to reformulate a 2nd-order IVP into a system of two 1st-order IVPS
  - Look at an example
  - Consider a 5th-order IVP
  - Reformulate that IVP into a system of five 1st-order IVPS

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2nd-order initial-value problems

- You have seen 2nd-order initial-value problems (IVPs) written in the form:

$$a_2(t)\,y^{(2)}(t)+a_1(t)\,y^{(1)}(t)+a_0(t)\,y(t)=g(t)$$
$$y(t_0)=y_0$$
$$y^{(1)}(t_0)=y_0^{(1)}$$

- Based on the implicit function theorem,
  we can always write a higher-order IVP as

$$y^{(n)}(t)=f\left(t,y(t),y^{(1)}(t),\ldots,y^{(n-1)}(t)\right)$$
$$y(t_0)=y_0$$
$$y^{(1)}(t_0)=y_0^{(1)}$$
$$\vdots$$
$$y^{(n-1)}(t_0)=y_0^{(n-1)}$$

2nd-order initial-value problems

- Consequently, a second-order IVP can be written as

$$y^{(2)}(t)=f\left(t,y(t),y^{(1)}(t)\right)$$
$$y(t_0)=y_0$$
$$y^{(1)}(t_0)=y_0^{(1)}$$

- Now, we don't know $y(t)$, nor do we know $y^{(1)}(t)$
  - Thus, suppose we define

$$w_0(t)=y(t) \qquad\qquad w_1(t)=y^{(1)}(t)$$

  - Note therefore that

$$w_0^{(1)}(t)=y^{(1)}(t)\;=w_1(t) \qquad\qquad w_0(t_0)=y(t_0)=y_0$$
$$w_1^{(1)}(t)=y^{(2)}(t)\;=f\left(t,y(t),y^{(1)}(t)\right) \qquad w_1(t_0)=y^{(1)}(t_0)=y_0^{(1)}$$
$$=f\left(t,w_0(t),w_1(t)\right)$$

2nd-order initial-value problems

- Consequently, a second-order IVP can be written as a system of two 1st-order IVPs

$$y^{(2)}(t) = f\left(t, y(t), y^{(1)}(t)\right) \qquad w_0^{(1)}(t) = w_1(t) \qquad w_0(t_0) = y_0$$

$$y(t_0) = y_0 \qquad w_1^{(1)}(t) = f\left(t, w_0(t), w_1(t)\right) \qquad w_1(t_0) = y_0^{(1)}$$

$$y^{(1)}(t_0) = y_0^{(1)}$$

- In the last topic, we saw how we can write this with vectors

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} \quad \mathbf{w}^{(1)}(t) = \begin{pmatrix} w_1(t) \\ f\left(t, w_0(t), w_1(t)\right) \end{pmatrix} \quad \mathbf{w}(t_0) = \begin{pmatrix} y_0 \\ y_0^{(1)} \end{pmatrix}$$

2nd-order initial-value problems

- For example,

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} \quad \mathbf{w}^{(1)}(t) = \begin{pmatrix} w_1(t) \\ f\left(t, w_0(t), w_1(t)\right) \end{pmatrix} \quad \mathbf{w}(t_0) = \begin{pmatrix} y_0 \\ y_0^{(1)} \end{pmatrix}$$

- In the last topic, we saw how we can write this with vectors

```
vector F( double t, vector w ) {
    return vector{ 2, (double[]){ w(1),
                                  f( t, w ) } };
}

// y"(t) = 2y'(t) - y(t) + sin(t)
double f( double t, vector w ) {
    return 2*w(1) - w(0) + sin(t);
}
```

2^{nd}-order initial-value problems

- Consider the following:

$$y^{(2)}(t) + 2y^{(1)}(t) + y(t) = \sin(t)$$
$$y(0) = 1$$
$$y^{(1)}(0) = -2$$

- In the last topic, we saw how we can write this with vectors

```
vector F( double t, vector w ) {
    return vector{ 2, (double[]){ w(1),
                                  -2*w(1) - w(0) + sin(t) } };
}
```

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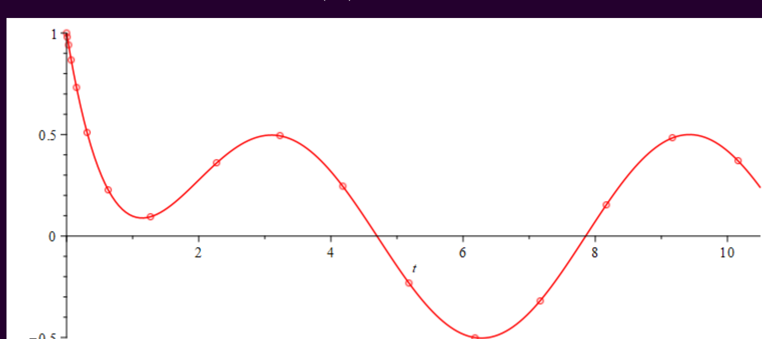
2^{nd}-order initial-value problems

- The solution and our approximation is shown here:

$$y^{(2)}(t) + 2y^{(1)}(t) + y(t) = \sin(t) \qquad y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}te^{-t} - \frac{1}{2}\cos(t)$$
$$y(t_0) = 1$$
$$y^{(1)}(t_0) = -2$$



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2^{nd}-order initial-value problems

- The output of the Dormand-Prince method is:

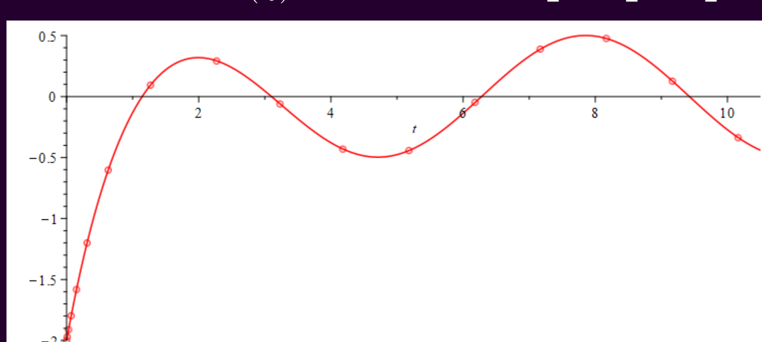
```
0                  < 1                 -2>
0.01               < 0.9801495012466744 -1.970149501662508  >
0.03               < 0.9413366004450955 -1.911336633992595  >
0.07               < 0.8671814460390597 -1.797182432449922  >
0.15               < 0.7321233276597681 -1.582143788605167  >
0.31               < 0.5103193853011617 -1.200680330337188  >
0.63               < 0.2271084350598108 -0.6028457398035203 >
1.27               < 0.09484159082902341 0.09414765884974807>
2.27               < 0.360724134043294   0.29190279275873   >
3.22870270906355   < 0.4949570390741321 -0.06046514781695389>
4.180454048644937  < 0.2454653632568291 -0.4306789648656997 >
5.180454048644937  <-0.2319093306500261 -0.4427291816128475 >
6.180454048644937  <-0.5018483904660588 -0.04756916700176944>
7.167318861458502  <-0.3196352998086384  0.3892455074943849 >
8.167318861458501  < 0.1533886469424242  0.4763189756994295 >
9.167318861458501  < 0.4843038344873439  0.1262539232541798 >
10.16224322948684  < 0.3711291770716665 -0.3375822706119238 >
```


2^{nd}-order initial-value problems

- The solution and our approximation is shown here:

$$y^{(2)}(t) + 2y^{(1)}(t) + y(t) = \sin(t)$$
$$y(t_0) = 1$$
$$y^{(1)}(t_0) = -2$$

$$y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}te^{-t} - \frac{1}{2}\cos(t)$$
$$y^{(1)}(t) = -\frac{3}{2}e^{-t} - \frac{1}{2}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{2}\sin(t)$$



Higher-order initial-value problems

- The process is the same for a higher-order IVP:

$$12.5y^{(5)}(t)+17.5y^{(4)}(t)+132.625y^{(3)}(t)+128.95y^{(2)}(t)+136.25y^{(1)}(t)+91.304y(t)=\cos(t)$$

$$y(0)=1.0$$
$$y^{(1)}(0)=-0.8$$
$$y^{(2)}(0)=0.7$$
$$y^{(3)}(0)=0.6$$
$$y^{(4)}(0)=-0.5$$

- This is the function:

```
vector F( double t, vector w ) {
    return vector{ 5, (double[]){
        w(1), w(2), w(3), w(4),
        (cos(t) -  17.5  *w(4) - 132.625*w(3)
                - 128.95 *w(2) - 136.25 *w(1)
                -  91.304*w(0))/12.5
    } };
}
```


Higher-order initial-value problems

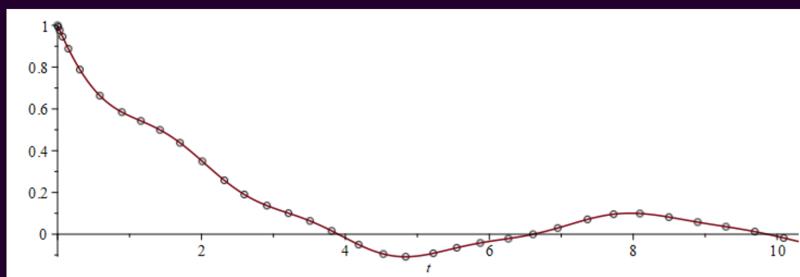
- This piece of code would print the appropriate output

```
int main() {
    auto result{ dp45(
        F, std::make_pair( 0.0, 10.0 ),
        vector{ 5, (double[]){ 1.0, -0.8, 0.7, 0.6, -0.5 } },
        std::make_pair( 1e-4, 1.0 ), 1e-2
    ) };

    std::cout << "n = " << std::get<0>( result ) << std::endl;

    for ( unsigned int k{0}; k <= std::get<0>( result ); ++k ) {
        std::cout << std::get<1>( result )[k] << ", "
                  << std::get<2>( result )[k] << std::endl;
    }

    return 0;
}
```

Approximating solutions to higher-order initial-value problems

Higher-order initial-value problems

- This is the solution to this 5^{th}-order IVP and our approximations



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Approximating solutions to higher-order initial-value problems

Summary

- Following this topic, you now
  - Have a better understanding of 2^{nd}- and higher-order initial-value problem
  - Understand how we can reformulate an n^{th}-order IVP into a system of n 1^{st}-order IVPs
  - Have seen examples of a 2^{nd}- and a 5^{th}-order IVP

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References

[1] https://en.wikipedia.org/wiki/Initial_value_problem

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Approximating solutions to higher-order initial-value problems

Acknowledgments

None so far.

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Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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